# Global Games With Strategic Substitutes*

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**Abstract**

This paper studies global games with strategic substitutes. Specifically, for a class of binary action, $N$-player games with strategic substitutes, we prove that each such game has a unique bayesian Nash equilibrium as incomplete information vanishes. The central assumption is payoff asymmetry among players in such a way that it provides sequentially overlapped dominance regions. Our result extends the global game literature which has been developed so far for games with strategic complementarities.

JEL codes: C72, D82, H41

# 1 Introduction

In general, game-theoretic models are developed under the assumption that the rational behavior of the players and the structure of the game are common knowledge. Since these assumptions might be too stringent for modeling real-life players, it is important to know whether the prediction of a game substantially changes in comparison to the predictions of a slightly altered version of the same game[1]. If indeed it turns out that only certain of the game's equilibria survive this "robustness check" then we may reasonably refine our prediction of what happens in such games.

This paper examines the dual issues of equilibrium selection and robustness in a class of games with strategic substitutes. These are games in which each player's marginal payoff from increasing her own action is decreasing in the other players' actions. The standard example is the game of voluntary contribution toward a public good. The equilibria exhibit a classical free rider problem: an individual is less willing to contribute the larger is the total contribution of others. If one's contribution is an indivisible choice such as a unit of time or effort, then voluntary contribution games typically exhibit multiple Nash equilibria, each corresponding to a distinct configuration of contributors and non-contributors.

To examine equilibrium selection in games such as these, we follow the *Global Games* approach pioneered by Carlsson and van Damme (1993).[2] The idea in this approach is to examine Nash equilibria as a limit of equilibria of payoff-perturbed games. More formally, suppose $G$ is a standard game of complete information where the payoffs depend on a parameter $x \in \mathbb{R}$, and also suppose that for some subset of the parameter $x$, $G$ has a strict Nash equilibrium. Rather than observing the parameter $x$, suppose instead that each player observes a private signal $x_i = x + \sigma\varepsilon_i$ where $\sigma > 0$ is a scale factor and $\varepsilon_i$ is a random variable with density $\phi$. Denote this "perturbed game" by $\mathrm{G}(\sigma)$, and let NE(G) and BNE($\mathrm{G}(\sigma)$) denote the sets of Nash and Bayesian Nash equilibria of the unperturbed and perturbed games, respectively. Equilibrium selection is obtained when point-wise on $x$, $\lim_{\sigma\to 0} \mathrm{BNE}(\mathrm{G}(\sigma))$ is a strict subset of NE(G).

Carlsson and van Damme (1993) show, in fact, that for two-player, two-action games, this limit comprises a single equilibrium profile. Moreover, this equilibrium profile is obtained through iterated

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[1] Examples in this direction are the seminal contributions of Harsanyi's games with randomly disturbed payoffs (Harsanyi, 1973), and Selten' concept of trembling hand perfection (Selten, 1975).

[2] For an excellent description and survey of the ensuing literature see Morris and Shin (2003).

deletion of strictly dominated strategies. Roughly, the deletion requires that, for each player and for each action of that player, there are certain extreme values of the parameter, $x$, for which that action is strictly dominant. Even if these values carry very little probability weight, the players can use signals close to these "dominance regions" to rule out certain types of behavior of others. Hence, the iterative deletion proceeds.

These results have been extended by Frankel, Morris, and Pauzner (2003) for games with many players and many actions. However, existing results in this literature are typically limited to the case of strategic complementarities (and some other technical assumptions). This strong result is very useful for many games such as bank run models (Goldstein and Pauzner, 2005), currency crises games (Morris and Shin, 1998), etc.

Yet, there is a wide class of games where this condition is not satisfied. The voluntary contribution to the provision of a public good mentioned above, is one example. Of course in the two player case, the game can be represented as a game of strategic complements by just reordering the set of actions. However, in games with more than two players, the analysis has not been extended to games of strategic substitutes.

The key insight in the present paper is to show how global games ideas can be applied to certain games of strategic substitutes when the players' payoffs display a certain, commonly known asymmetry. Specifically, for a class of binary actions games, we assume that there exists an ordering of players such that each player's dominance region is an arbitrary displacement to the right of the "previous" player's dominance region, i.e. the values of $x$ at which some player's upper dominance region begins and at which her lower dominance region finishes are strictly higher (lower) compared to those of any lower (higher) player. Under these assumptions and some other technical properties, the main result of the paper proves that there exists a unique equilibrium profile. Specifically, we show that as the noise goes to zero, a process of iterated elimination of *conditionally* dominated strategies converges to a single profile of *switching strategies*. In such a profile, each player has a threshold, cutoff signal above which she takes the higher (contributing) action, and below which the lower (non-contributing) action is taken. A very important characteristic of this profile is that each player has a different cutoff point. Interestingly, the order of these cutoff points is the same order that the players have. That is, the lower the player in the ordering, the smaller is her threshold. More precisely, the equilibrium predicts that the first player switches at the end of her lower dominance region, the last player switches at the beginning of her upper dominance region and all the other players have switching points in between these two. Intuitively, the equilibrium selected establishes that, if there are certain number of players choosing the contributing action, it must be the case that they are the lowest according to the players' order, conditional on the value of the parameter. Therefore, depending on the specific payoff structure of the game, the equilibrium profile structure might play an interesting role from an efficiency point of view. The result suggests that common knowledge of the order of players and global games structure are sufficient conditions to select not only a unique but also an ordered equilibrium.

As an introductory example, in Section 2, we present a game of public good provision, where all the assumptions are satisfied. The main result for this game is that for general distributional properties of the signal noise, there exists a unique strategy profile played in equilibrium. This profile induces an efficient provision of the public good, and the contributions come from the lowest cost contributors. This result suggests that inefficient contribution equilibria survive only under a pair of stringent assumptions: common knowledge of the fundamentals, and perfect symmetry in the players' characteristics. In sections 3 and 4 we present the general framework and establish our main result. In section 5 we develop the main steps of the proof and finally, in section 6 we presents the conclusions. Proofs of propositions and lemmas are relegated to the appendix.

## 2 Example: Public Good Provision

In many collective action problems multiple Nash equilibria may exist, each corresponding to a different configuration of contributors. Many of these equilibria are inefficient since individuals with a higher marginal cost of contributing end up contributing disproportionately. Here, we prove a result that suggests that these inefficient Nash equilibria are not robust.

We develop a binary action game of incomplete information in which the mechanism for public good provision utilizes both government and voluntary contributions. In particular, to fund a public good, a government pledges "seed money" which must be augmented by funds from private contributors.

Each contributor, upon receiving a private signal of the amount of this pledge, then chooses whether to contribute. Players have costs of contributing.

## 2.1 The Game

Consider the following game: a social planner decides to provide a public good $G$, requiring society's contribution. The society is composed by 2 different individuals indexed by $i \in \{1, 2\}$. Each player $i$ has to decide whether to contribute, choosing an indivisible action $a_i$ from the binary set $A_i = \{1 = \text{contribute}, 0 = \text{not contribute}\}$.

Let $Y(x, n) = x^{1-\alpha} n^{\alpha}$ denote the public good technology, where $x \in \left[0, \bar{X}\right]$ is the government contribution and $n$ is the number of people who decide to contribute and $0 < \alpha < 1$. Without loss of generality we can characterize the payoffs as follows: if player $i$ chooses to contribute, she has to provide an effort (contribution) $c_i > 0$, and receives a utility $Y(x, n + 1) - c_i$, where $n = 1$ if the other player contributes and $n = 0$ if not. On the other hand, if the same player chooses not to contribute (free ride), she will receive a utility $Y(x, n)$. Let

$$\Delta\pi_i(x, n) := Y(x, n + 1) - c_i - Y(x, n) = x^{1-\alpha} \left((n + 1)^{\alpha} - n^{\alpha}\right) - c_i$$

be player $i$'s net payoff from contributing, given a value $n \in \{0, 1\}$.

Notice that although this is a 2x2 game, its generalization to more players is a game of strategic substitutes. In general, the greater the other players' strategy, the smaller is player $i$'s incentive to increase her strategy. Also, we have that the higher the social planner's contribution, the greater the player's incentive to contribute. Finally, note that for sufficiently high (low) values of the social planner's contribution, player $i$ will always (never) contribute, i.e. (not) contributing is a strictly dominant strategy. We call these ranges of values *dominance regions*.

For a given $x$ we can represent this game in the following normal form:

| | | Player 2: 1 | Player 2: 0 |
|---|---|---|---|
| Player 1 | 1 | $x^{1-\alpha}2^{\alpha} - c_1$, $x^{1-\alpha}2^{\alpha} - c_2$ | $x^{1-\alpha} - c_1$, $x^{1-\alpha}$ |
| | 0 | $x^{1-\alpha}$, $x^{1-\alpha} - c_2$ | $0$, $0$ |

First suppose $c_1 = c_2 = c$, the symmetric case. Then both players have the same payoff function. Consider the following equations and their solutions $\underline{k}$ and $\bar{k}$:

$$\Delta\pi_i(\underline{k}, 0) = 0 \qquad \Longrightarrow \qquad \underline{k} = c^{\frac{1}{1-\alpha}}$$

and

$$\Delta\pi_i\left(\bar{k}, 1\right) = 0 \qquad \Longrightarrow \qquad \bar{k} = \left(\frac{c}{2^{\alpha} - 1}\right)^{\frac{1}{1-\alpha}}.$$

Note that if $x < \underline{k}$, then action 0 is strictly dominant for both players and if $x > \bar{k}$ then action 1 is strictly dominant for both players. The values $\underline{k}$ and $\bar{k}$ define the lower and upper *dominance regions* shown in Figure 1. Thus, if $x > \bar{k}$ (resp. $x < \underline{k}$) then the parameter $x$ is in the upper (resp. lower) dominance region for both players.

The set of Nash equilibria as a function of $x$ has the following structure:

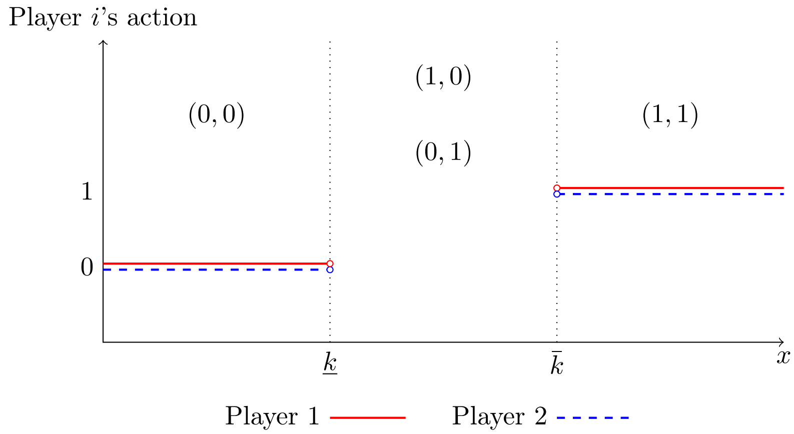


Figure 1: Dominance regions and equilibria in the Symmetric Case.

- For values of $x$ in the dominance regions, both players choose the dominant action. In Figure 1 the solid lines represent the dominant action of player 1 as a function of $x$, and dashed lines represent the dominant action of player 2. Therefore in each dominance region there exists a unique equilibrium: $(1, 1)$ in the upper dominance region and $(0, 0)$ in the lower dominance region.
- If $x$ takes values in the interval $\left]\underline{k}, \bar{k}\right[$ there are two pure strategy Nash equilibria. In these two equilibria one player chooses to contribute and the other one chooses not to contribute. Then, the equilibrium profiles are $(1, 0)$ and $(0, 1)$.

### 2.2 Incomplete Information

Suppose now that the game is characterized by incomplete information in the payoff structure. Instead of observing the actual value of the social planner's contribution $x$, each player just observes a private signal $x_i$, which contains diffuse information about $x$. The signal has the following structure: $x_i = x + \sigma\varepsilon_i$, where $\sigma > 0$ is a scale factor, $x$ is drawn from $\left[\underline{X}, \bar{X}\right]$, where $\underline{X} > 0$ and $\bar{X}$ is sufficiently large,[3] with uniform density and each $\varepsilon_i$ is randomly selected independently of $x$ on the interval $[-1, 1]$. We assume that the random vector $\varepsilon = (\varepsilon_1, \varepsilon_2)$ admits a continuous density.

In this context of incomplete information, a Bayesian pure strategy for player $i$, is a function $s_i : \left[\underline{X} - \sigma, \bar{X} + \sigma\right] \to A_i$. A pure strategy profile is a vector $s = (s_1, s_2)$.

Calling $\mathrm{G}_{\{1,2\}}(\sigma)$ this game of incomplete information, let $\mathrm{BNE}\big(\mathrm{G}_{\{1,2\}}(\sigma)\big)$ be the set of Bayesian Nash equilibria of $\mathrm{G}_{\{1,2\}}(\sigma)$.

At a first glance the incomplete information game $\mathrm{G}_{\{1,2\}}(\sigma)$ has a similar structure to the class of games studied in Carlsson and van Damme (1993). One may wish to apply their Theorem to obtain an equilibrium selection result in $\mathrm{G}_{\{1,2\}}(\sigma)$. However, the Nash equilibria structure described above suggests two important observations. First, the Carlsson and van Damme global game equilibrium selection result cannot be applied to this game because it requires that a selected equilibrium be a unique Nash equilibrium for some subset of values of the exogenous parameter ($x$ in this case) (Carlsson and van Damme, 1993). Moreover, the selected equilibrium should be the risk dominant equilibrium. In this game, when we have multiplicity, none of the equilibria profiles ($(0, 1)$ and $(1, 0)$) is a unique Nash equilibrium for some value of $x$ nor we have a risk dominant equilibrium (since $c_1 = c_2$). Second, for $x \in \left]\underline{k}, \bar{k}\right[$ this symmetric case not only entails multiplicity of equilibria, but also that each of the equilibria has an asymmetric structure where just one of the players contributes. This suggests that asymmetry may play an important role in any equilibrium selection attempt.

Let us now introduce asymmetry in the payoff structure of the game by assuming that $c_2 > c_1 > 0$.

[3] Such that $\bar{X}$ is contained in every player's upper dominance region. We care about games in which $\sigma < \underline{X}$ so that $x_i > 0$ in all the support.

This generates different dominance regions for each player, determined by the values:

$$\underline{k}_i = c_i^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad \bar{k}_i = \left(\frac{c_i}{2^\alpha - 1}\right)^{\frac{1}{1-\alpha}}, \qquad i \in \{1,2\}.$$

Note that $\underline{k}_1 < \underline{k}_2$ and $\bar{k}_1 < \bar{k}_2$.

In Figure 2 we can observe that the asymmetry generates the overlapping of the dominance regions. A very important consequence of this, is the generation of a subset of values of $x$ where the profile $(1,0)$ is the unique equilibrium. Indeed, if $x \in \left]\underline{k}_1, \underline{k}_2\right[ \cup \left]\bar{k}_1, \bar{k}_2\right[$, the unique equilibrium of the game parametrised on $x$ is $(1,0)$. Moreover, $(1,0)$ is the risk dominant equilibrium. This enables us to apply the Carlsson and van Damme global game equilibrium selection result.

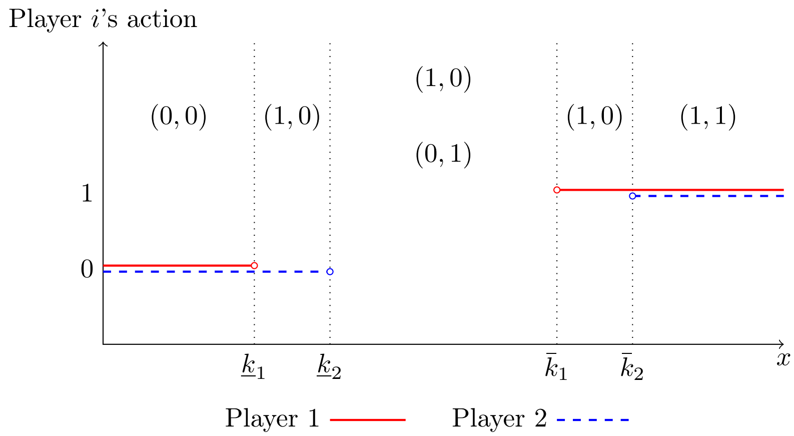


Figure 2: Dominance regions and equilibria in the Asymmetric Case.

Define $s^*$ as a particular profile of *switching strategies*, such that player 1 and player 2 switch from action 0 to action 1 at the cutoff points $\underline{k}_1$ and $\bar{k}_2$ respectively:

$$s_1^*(x_1) = \begin{cases} 0 & \text{if } x_1 < \underline{k}_1 \\ 1 & \text{if } x_1 > \underline{k}_1 \end{cases} \qquad s_2^*(x_2) = \begin{cases} 0 & \text{if } x_2 < \bar{k}_2 \\ 1 & \text{if } x_2 > \bar{k}_2 \end{cases}.$$

For this game we can restate the Carlsson and van Damme (1993) result, in terms of Theorem 1 in Frankel, Morris, and Pauzner (2003), in the following proposition:

**Proposition 1.** *Consider the game* $\mathrm{G}_{\{1,2\}}(\sigma)$ *introduced above. The strategy profile* $s^*$ *is the unique strategy profile that survives iterated deletion of strictly dominated strategies for a sufficiently small amount of noise.*[4]

There are two results in Proposition 1: uniqueness of equilibrium and dominance solvability, when $\sigma \to 0$. Our interest is on uniqueness of equilibrium in games of strategic substitutes with more than 2 players. As we will see below, starting from the setting of Frankel et al. (2003), but with two actions and strategic substitutes, it is not possible to obtain dominance solvability. However, introducing certain natural player asymmetry it is possible to prove uniqueness of equilibrium when $\sigma \to 0$. Thus, uniqueness of equilibrium in Proposition 1 is a direct consequence of our Theorem stated in Section 4.

Figure 3 shows the structure of the equilibrium profile $s^*$: player 1 switches from not contributing to contributing at $\underline{k}_1$; and player 2 switches at $\bar{k}_2$. It is important to notice that this strategy profile induces an efficient provision of the public good, and that the contributions come from the lowest cost contributors. The result suggests that inefficient contribution equilibria survive only under a pair of stringent assumptions: common knowledge of the fundamentals, and perfect symmetry in the players' characteristics.

[4]This result may be obtained by applying the Theorem on page 996 in Carlsson and van Damme (1993) or Theorem 1 in Frankel et al. (2003). Note that being a two 2x2 game, the game is both of strategic complementarities and strategic substitutes.

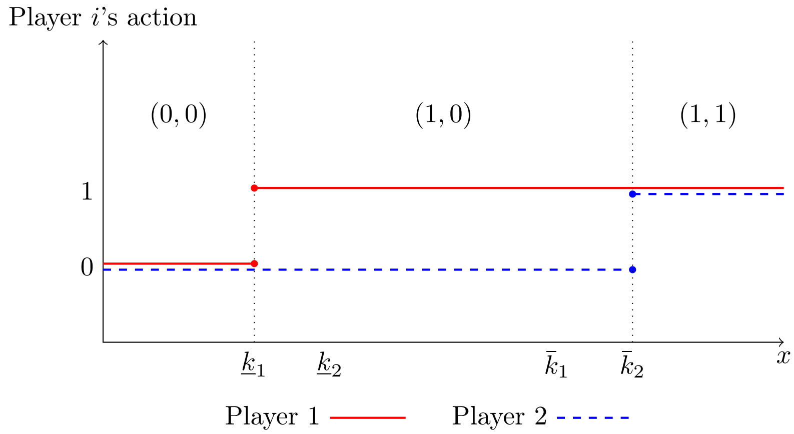


Figure 3: Global game equilibrium and equilibrium selection: two players case.

The existence of overlapped dominance regions allowed us to select a particular equilibrium. This suggests that generalizing this payoff structure, under the global games approach, we can prove the existence of a unique equilibrium in a class of global games with strategic substitutes.

In the next sections we develop a more general framework and we state and prove our main result: the existence of a unique equilibrium profile in a certain class of global games with strategic substitutes.

# 3 General Framework

Consider the following general setup for an $N$ person game $\mathrm{G}_{\mathcal{N}}(x)$ parametrized by the exogenous variable $x$, which takes values in the interval $\left[\underline{X}, \overline{X}\right] \subset \mathbb{R}$. There are $N$ anonymous players indexed by $i \in \mathcal{N} = \{1, \ldots, N\}$ and each player has a binary set of actions $A_i = \{0, 1\}$.[5] Player $i$'s payoff function is $u_i(a_i, a_{-i}, x) = \pi_i\left(a_i, \sum_{j \neq i} a_j, x\right)$. Where $\pi_i : \{0,1\} \times \{0, ..., N-1\} \times \left[\underline{X}, \overline{X}\right] \to \mathbb{R}$ is an auxiliary function that depends on other players' actions through the number of players (other than $i$) that are choosing action 1.

Let us define $\Delta\pi_i(n, x) = \pi_i(1, n, x) - \pi_i(0, n, x)$ as player $i$'s payoff difference when she is choosing action 1 rather than action 0. We consider that the payoff structure of the class of games $\mathrm{G}_{\mathcal{N}}(x)$, $x \in \left[\underline{X}, \overline{X}\right]$, satisfies the following assumptions:

A1. **Strategic Substitutes (SS)** Conditional on the value of $x$ the greater the other players' action profile, the smaller is player $i$'s incentive to choose the higher action:

If $n > n'$, $\Delta\pi_i(n, x) < \Delta\pi_i(n', x) \quad \forall\ x.$

A2. **Continuity (C)** $\pi_i(a_i, n, x)$ is a continuous function of $x$.

A3. **Monotonicity (M)** The greater the value of the exogenous variable $x$, the greater the player $i$'s incentive to choose the higher action:

$\exists\ c > 0$ s.t. if $x, x' \in \left[\underline{X}, \overline{X}\right]$ and $x \geq x'$, then $\Delta\pi_i(n, x) - \Delta\pi_i(n, x') \geq c\,(x - x')$, $\forall\ n$.

A4. **Indifference Points (IP)** If other players are choosing identical actions, there exists a value of $x$ such that player $i$ is indifferent between the two actions:[6]

$\forall\ i\ \exists\ \underline{k}_i > \underline{X}$ s.t. $\Delta\pi_i(0, \underline{k}_i) = 0$ and $\exists\ \bar{k}_i$ s.t. $\overline{X} > \bar{k}_i$ s.t. $\Delta\pi_i\left(N-1, \bar{k}_i\right) = 0$.

A5. **Players Order (PO)** There exists a players order $\{1, ..., N\}$ such that if $j > i$, then if both players observe the same value of $x$ and face the same $n$, player $j$ has less incentive than $i$ to

[5] We will also refer to $a_i = 0$ as the "lower" action and $a_i = 1$ as the "higher" action.

[6] Note that because of A3 (M) these indifference values are unique, and A1 (SS) along with A3 (M) imply that $\bar{k}_i > \underline{k}_i$.

pick the higher action.[7] That is,[8]

$$j > i \iff \Delta\pi_i(n,x) - \Delta\pi_j(n,x) > 0, \forall n, \forall x.$$

An important remark is that assumptions A1 (SS), A3 (M) and A4 (IP) provide sufficient conditions for the existence of dominance regions, along which each action is strictly dominant, i.e. $\forall\ x < \underline{k}_i$, $\Delta\pi_i(n,x) < 0$ and $\forall\ x > \bar{k}_i$, $\Delta\pi_i(n,x) > 0\ \forall n$.

Additionally, these assumptions allow us to state a more general single crossing property, which will help to characterize the equilibrium profile:

**Lemma 1.** *For all $i \in \mathcal{N}$ and for all $n \in \{0,\dots,N-1\}$ there exists a unique $k_i(n) \in [\underline{X},\overline{X}]$ solving $\Delta\pi_i(n,k_i(n)) = 0$. Moreover, $\Delta\pi_i(n,x) < 0$, $\forall\ x < k_i(n)$; and $\Delta\pi_i(n,x) > 0$, $\forall\ x > k_i(n)$.*

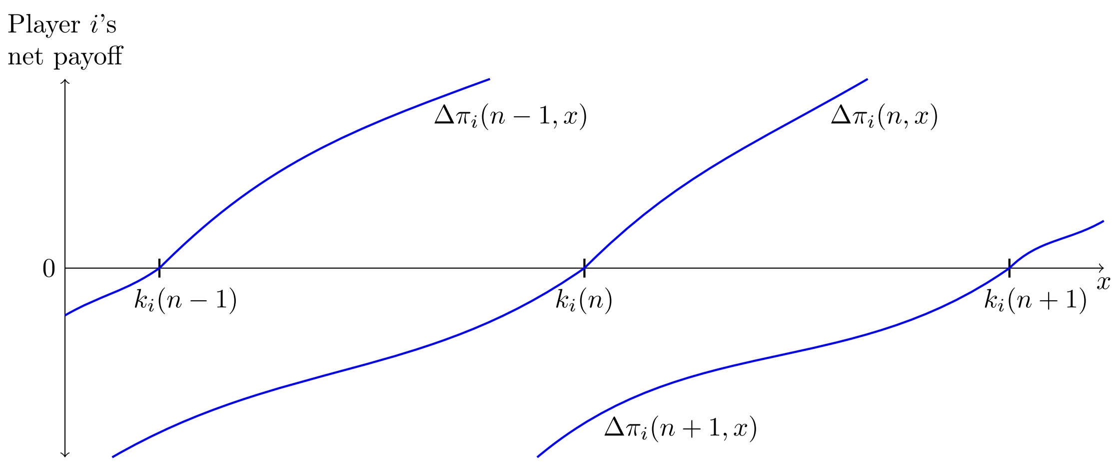


Figure 4: Player $i$'s payoffs dependence on $x$.

In Figure 4, we can observe how player $i$'s payoffs depend on $x$. From Lemma 1 we know that for each $n$ there exists a unique $k_i(n)$ such that player $i$ is indifferent between the two actions; and that given $n$, player $i$'s best response is to switch from the lower action to the higher action at a unique value of the signal. Note then that $\underline{k}_i = k_i(0)$ and $\bar{k}_i = k_i(N-1)$. Given assumption A3 (M) we know that the net payoff function is monotonic in $x$ and by assumption A1 (SS) we know that for different $n$ the net payoff functions do not intersect each other.

Additionally A5 (PO) implies that if $j > i$ then $k_j(n) > k_i(n)$ for every $n \in \{0,\dots,N-1\}$. In particular $\underline{k}_j > \underline{k}_i$ and $\bar{k}_j > \bar{k}_i$.[9] In Figure 5, for a three player case, we can observe a direct consequence of this assumption: sequentially overlapped dominance regions. Therefore assumption A5 (PO) provides the necessary asymmetry in the game.

The last important remark about the assumptions is contained in the following lemma:

**Lemma 2.** *There exists a value $\sigma_0 > 0$ such that $\forall\ \sigma \in (0,\sigma_0)$, if $j > i$ and $x_j - x_i \leq \sigma$, then $\Delta\pi_i(n,x_i) - \Delta\pi_j(n,x_j) > 0\ \forall\ n$.*

From assumption A5 (PO), we know that if two players face the same strategy profile and the same value of $x$, the "greater" player will get a lower net payoff. This lemma states that this is still true even when they face different values of $x$, such that their difference is less than $\sigma_0$.

## 3.1 Incomplete Information

We now endow the class of games $G_{\mathcal{N}}(x)$, $x \in [\underline{X},\overline{X}]$, with incomplete information in the payoff structure. Instead of observing the actual value of $x$, each player just observes a private signal $x_i$, which contains diffuse information about $x$.

[7] This condition provides players' asymmetry.

[8] Note that from compactness of $[\underline{X},\bar{X}]$, this implies that $\exists\ \alpha > 0$ such that $\Delta\pi_i(n,x) - \Delta\pi_j(n,x) > \alpha$, for all $j > i$.

[9] Without loss of generality in the analysis we will assume the case where $\underline{k}_N < \bar{k}_1$, excluding the trivial situations where $\underline{k}_N > \bar{k}_1$, i.e. player $N$'s lower dominance region does not overlap player 1's upper dominance region.

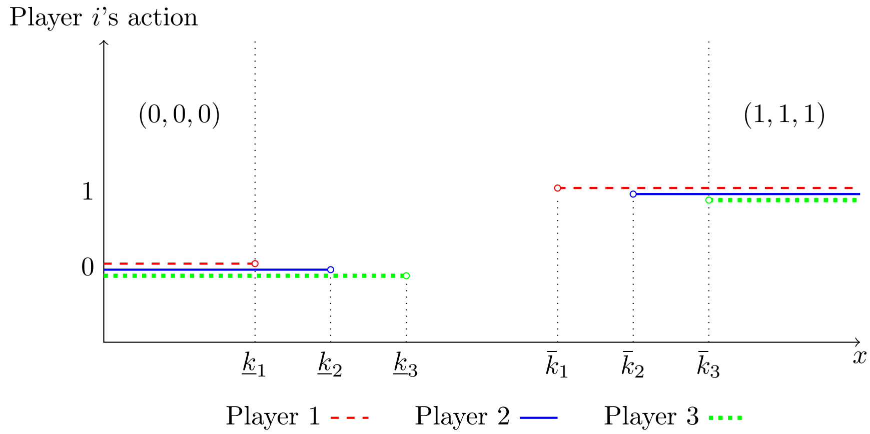


Figure 5: Overlapped Dominance Regions: Three Players Case.

The signal has the following structure: $x_i = x + \sigma\varepsilon_i$, where $\sigma > 0$ is a scale factor, $x$ is drawn from the interval $\left[\underline{X}, \overline{X}\right]$ with uniform density and each $\varepsilon_i$ is randomly selected independently of $x$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In this context signals $x_i$ belong to the set $X(\sigma) = \left[\underline{X} - \frac{1}{2}\sigma, \overline{X} + \frac{1}{2}\sigma\right]$. We call this game of incomplete information $\mathrm{G}_{\mathcal{N}}(\sigma)$.

This general noise structure has been used in the global game literature, allowing the conditional distribution of the opponents signal to be modelled in a simple way, i.e. given a player's own signal, the conditional distribution of an opponent's signal $x_j$ admits a continuous density $f_\sigma$ and a cdf $F_\sigma$ with support in the interval $[x_i - \sigma, x_i + \sigma]$. Moreover this literature establishes a significant result: when the prior is uniform, players' posterior beliefs about the difference between their own observation and other players' observations are the same, i.e. $F_\sigma(x_i \mid x_j) = 1 - F_\sigma(x_j \mid x_i)$.[10] To obtain continuity of the cdf $F_\sigma$, in this paper we assume, following Carlsson and van Damme (1993), that the random vector $(\varepsilon_1, \ldots, \varepsilon_N)$ admits a continuous density.

### 3.2 Strategies and Equilibrium

A Bayesian pure strategy for a player $i$, is a function $s_i : X(\sigma) \to A_i$, i.e. conditional on receiving a signal $x_i$ player $i$ takes an action $s_i(x_i) \in \{0, 1\}$. A pure strategy profile is denoted as $s = (s_1, s_2, \ldots, s_N)$ and $s_i \in S_i$, the set of all functions from $X(\sigma)$ to $A_i$; equivalently we use the usual notation $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N) \in S_{-i}$.

In this context of incomplete information, player $i$'s payoff is characterized by her beliefs about her opponents strategies and the value of $x$. In general, if player $i$ is observing a signal $x_i$ and is facing a strategy $s_{-i}$ her expected payoff of choosing action $a_i$ is

$$\Pi_i(a_i, s_{-i}, x_i) := \int_{x_i - \frac{\sigma}{2}}^{x_i + \frac{\sigma}{2}} \int_{x_{-i}} \pi_i\left(a_i, \sum_{j \neq i} s_j(x_j)\,, x\right) dF_{(\sigma,-i)}(x_{-i} \mid x_i)\, dP_{\sigma,i}(x \mid x_i) \tag{1}$$

and her expected net gain of choosing action 1 instead of action 0 can be written as

$$\Delta\Pi_i(s_{-i}, x_i) := \int_{x_i - \frac{\sigma}{2}}^{x_i + \frac{\sigma}{2}} \int_{x_{-i}} \Delta\pi_i\left(\sum_{j \neq i} s_j(x_j)\,, x\right) dF_{(\sigma,-i)}(x_{-i} \mid x_i)\, dP_{\sigma,i}(x \mid x_i) \tag{2}$$

where $dP_{\sigma,i}(x \mid x_i)$ is the posterior belief over $x$ after receiving the signal $x_i$.

**Definition 1.** *A strategy profile $s^e$ is a* Bayesian Nash Equilibrium *of $\mathrm{G}_{\mathcal{N}}(\sigma)$ if for every player $i \in \mathcal{N}$ we have*

$$\Pi_i\big(s_i^e(x_i)\,, s_{-i}^e, x_i\big) \geq \Pi_i\big(a_i, s_{-i}^e, x_i\big) \qquad a_i \in \{0,1\} \text{ and } \forall\ x_i \in X(\sigma) \tag{3}$$

[10] This property holds approximately when $x$ is not distributed with uniform density but $\sigma$ is small, i.e. $F(x_i \mid x_j) \approx 1 - F(x_j \mid x_i)$ as $\sigma$ goes to zero. See details in Lemma 4.1 Carlsson and van Damme (1993).

We will denote by $\mathrm{BNE}(\mathrm{G}_{\mathcal{N}}(\sigma))$ the set of Bayesian Nash equilibria of $\mathrm{G}_{\mathcal{N}}(\sigma)$.

# 4 Main Result

A *switching strategy* of a player $i$ is a Bayesian pure strategy $s_i$ satisfying: $\exists\, y$ s.t.

$$s_i(x_i) = \begin{cases} 0 & \text{if } x_i < y \\ 1 & \text{if } x_i > y \end{cases} \tag{4}$$

Abusing notation, we write $\hat{s}_i(\cdot; y)$ to denote the switching strategy of player $i$ with switching threshold $y$.

Let $s^*$ be the profile such that each player is using a switching strategy $\hat{s}_i(\cdot; x_i^*)$ where the threshold $x_i^*$ solves the following equation:

$$\Delta\pi_i(i-1, x_i^*) = 0. \tag{5}$$

In this profile, player $i$ will switch from 0 to 1 at $x_i^*$, where $x_i^*$ is the indifference point, when she faces a strategy profile such that all the players "lower" than her play action 1 and all the "higher" players play action 0. From Lemma 1 we know that for all $i$, $x_i^* = k_i(i-1)$ not only exists, but it is also unique. We will prove that $s^*$ is the unique equilibrium profile of $\mathrm{G}_{\mathcal{N}}(\sigma)$ as $\sigma$ goes to 0.

Note that the strategy $s_i^*$ does not depend on the number of players. This is, the player that is in the $i$-th position (player $i$) determines her switching point depending only on her position in the set of players and not on how many players interact with her.[11]

We now formally state our main result.

**Theorem.** *The profile $s^*$ is the essentially unique strategy profile of $\mathrm{G}_{\mathcal{N}}(\sigma)$ as $\sigma \to 0$. More precisely, if $s \in \mathrm{BNE}(\mathrm{G}_{\mathcal{N}}(\sigma))$, then each player $i \in \mathcal{N}$ and for almost all $x_i \in X(\sigma)$ we have $\lim_{\sigma\to 0} s_i(x) = s_i^*(x)$.*

This result allows us to analyze a wide class of games of strategic substitutes where multiplicity is a problem, extending the global game literature.

In particular, this theorem generalizes the analysis and conclusion developed in the public good example of section 2; now, lower cost players are represented by a "higher" position in the players order (according to A5 (PO)), and they will switch between the actions at a higher threshold.

As an example, in Figures 6 and 7 we show the equilibria and the global game equilibrium selection respectively in a three players case. Figure 6 depicts the type of equilibria depending on the value of $x$ and the dominance regions for each player, while in Figure 7 we show the selected equilibrium profile in the global game. The strategy profile in equilibrium shows the higher player switching at the beginning of her upper dominance region, $x_3^* = \overline{k}_3$. The lower player switches at the end of her lower dominance region $x_1^* = \underline{k}_1$, and player 2 switches at $x_2^* = k_2(1)$ where $\underline{k}_1 < x_2^* < \overline{k}_3$.

# 5 Proof of the Theorem

In this section we develop the proof of the Theorem stated in Section 4. We will argue that the profile $s^*$ is the unique profile surviving a *particular* process of iterated deletion of strategies. The sets of surviving strategy profiles are not the standard undominated sets used to define iteratively undominated strategies. Instead these sets are defined by an alternative process that eliminates profiles that are not part of any equilibrium. These strategies are strictly dominated when we restrict ourselves to considering some subset of others players' strategies that are "potentially" part of some equilibrium profile. We call these sets the *conditionally iteratively undominated sets.*[12] We will prove that $s^*$ is in fact an equilibrium and that this process does not rule out any Bayesian Nash equilibria. We then

[11] According to the definition of $s^*$ in an $N$-player game, player 1 switches action at $\underline{k}_1 = k_1(0)$ and player $N$ at $\bar{k}_N = k_N(N-1)$. Therefore, if we consider a game with $N+1$ players, then the switching point of player $N$ is $x_N^* = k_N(N-1) \neq \bar{k}_N = k_N(N)$. When we pass from a game with $N$ players to a game with $N+1$ players, what changes is the limits of the upper dominance regions (they are $k_i(N-1)$ in the $N$-player game and $k_i(N)$ in the $N+1$-player game).

[12] Since the elimination proceeds upon players receiving the signal, then formally these sets contain strategies that are *interim* strictly undominated.

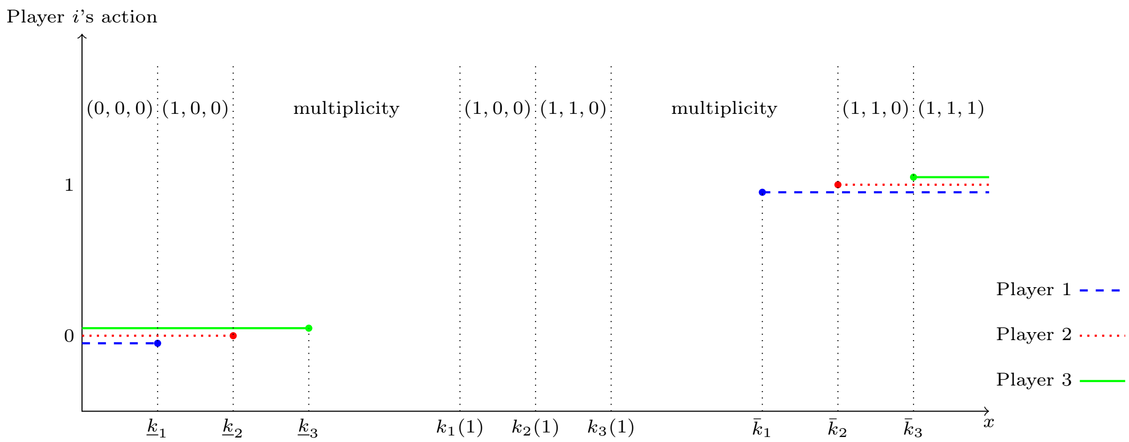


Figure 6: Dominance regions and equilibria.

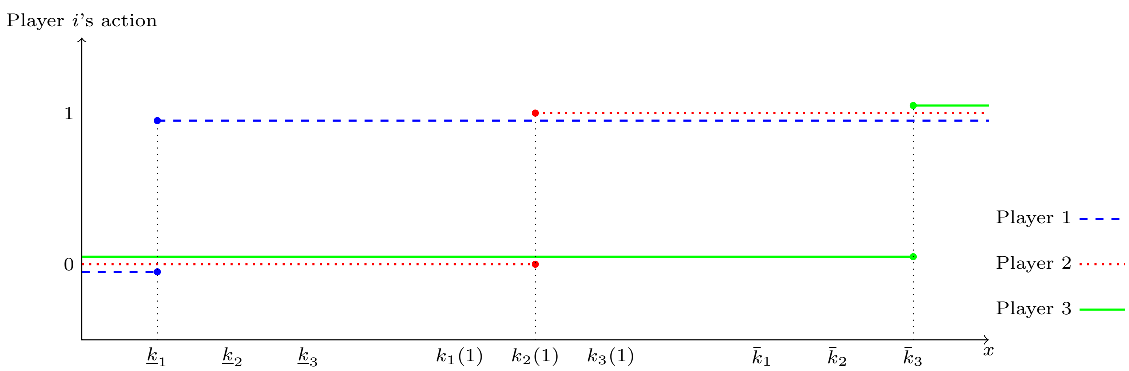


Figure 7: Equilibrium Selection: Three Players Case

proceed to show that under our assumptions the strategy profile surviving the iterated deletion is unique.

In order to define the process of elimination, we first need to introduce some concepts and notation.

## 5.1 Previous definitions

In this section we define the concept of *parametrized strategy set* and the notion of *extremal profiles*.

**Definition 2** (Parametrized strategy set)**.** *For each player $i \in \mathcal{N}$ and a number $z \in \left[\underline{k}_i, \bar{k}_i\right]$ define the* parametrized (on $z$) strategy set *of player $i$, $S_i(z)$, as follows:*

$$S_i(z) := \left\{s_i \in S_i : \ s_i(x_i) = 0 \text{ if } x_i < \min\{z, x_i^*\} \text{ and } s_i(x_i) = 1 \text{ if } x_i \in (x_i^*, z) \cup \left(\bar{k}_i, \overline{X} + \sigma\right)\right\}$$

All strategies in $S_i(z)$ prescribe action 0 for signals less than the minimum between $z$ and $x_i^*$; and prescribe action 1 for signals in player $i$'s upper dominance region and for signals in the interval $(x_i^*, z)$, if not empty. In the interval $\left(z, \bar{k}_i\right)$ the strategies in $S_i(z)$ may take any value (see Figure 8).

For each $i \in \mathcal{N}$ let us define player $i$'s best response correspondence $\mathrm{BR}_i : S_{-i} \rightrightarrows S_i$ as:

$$\mathrm{BR}_i(s_{-i}) := \{s_i \in S_i : \Pi_i(s_i(x_i), s_{-i}, x_i) \geq \Pi_i(a_i, s_{-i}, x_i) \ \forall \ x_i \in X(\sigma) \quad \forall \ a_i \in A_i\}$$

**Definition 3** (Parametrized mutually best response strategy profiles set)**.** *For each player $i \in \mathcal{N}$ and a tuple of signal values $x \in \prod_j \left[\underline{k}_j, \bar{k}_j\right]$ define the* parametrized (on $x$) mutually best response strategy

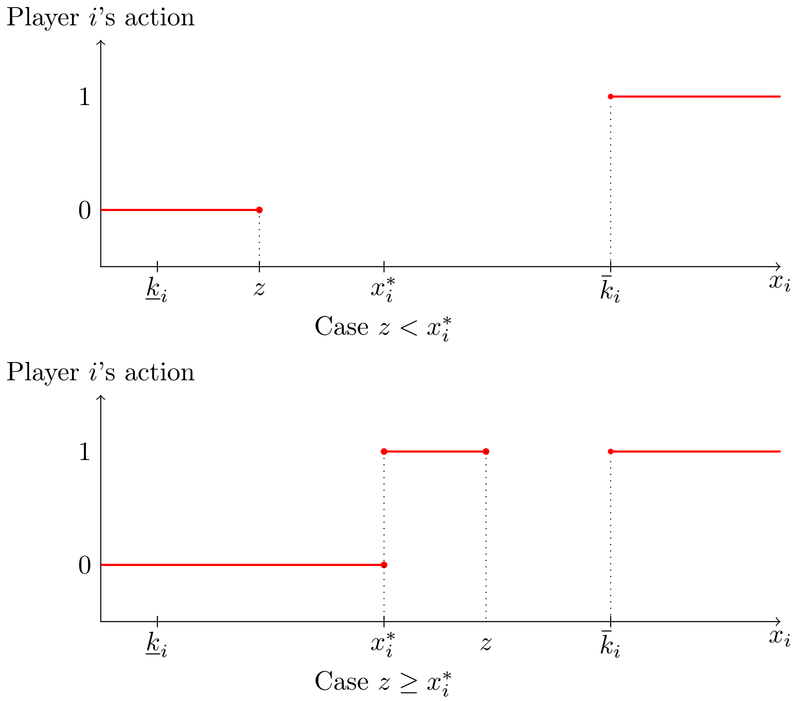


Figure 8: The set $S_i(z)$.

profiles set *of the rivals of player* $i$*,* $\mathcal{S}_{-i}(x)$*, as follows:*

$$\mathcal{S}_{-i}(x) := \left\{ s_{-i} \in \prod_{j \neq i} S_j(x_j) : \exists\ s_i \in S_i(x_i) \textit{ such that } s_j \in \mathrm{BR}_j(s_{-j})\ \forall\ j \neq i \right\}$$

For a given player $i \in \mathcal{N}$ the set $\mathcal{S}_{-i}(x)$ is the set of strategy profiles of all other players (i.e. excluding player $i$) that consist of strategies that are mutually best responses, for some strategy $s_i \in S_i(x_i)$ of player $i$.

Now we turn to the concept of extremal profiles. In general, for a payer $i \in \mathcal{N}$, given a set $M_{-i} \subseteq S_{-i}$ of strategy profiles of player $i$'s rivals, we may define two types of extremal profiles. The set $\overline{M}_{-i}$ contains the strategy profiles of the rivals of player $i$, that give player $i$ the most incentive to play action 1 at every $x_i$, while the set $\underline{M}_{-i}$ contains the strategy profiles of the rivals of player $i$, that give player $i$ the less incentive to play action 1 at every $x_i$. It is important to note that extremal profiles are determined by the set $M_{-i}$. More formally:[13]

**Definition 4** (Extremal Profiles)**.** *For each player* $i \in \mathcal{N}$ *and a set* $M_{-i} \subseteq S_{-i}$*, we define the sets of* Upper Extremal Profiles*,* $\overline{M}_{-i}$*, and* Lower Extremal Profiles*,* $\underline{M}_{-i}$*, faced by player* $i$ *associated to* $M_{-i}$ *as follows:*

$$\overline{M}_{-i} := \bigcap_{x_i \in [\underline{X}, \overline{X}]} \underset{s_{-i} \in M_{-i}}{\operatorname{argmax}} \Delta\Pi_i(s_{-i}, x_i)$$

$$\underline{M}_{-i} := \bigcap_{x_i \in [\underline{X}, \overline{X}]} \underset{s_{-i} \in M_{-i}}{\operatorname{argmin}} \Delta\Pi_i(s_{-i}, x_i)\,.$$

[13]Note that for a given $x_i \in [\underline{X}, \overline{X}]$, the extremal strategy of every rival of player $i$ for which she is uncertain about their play in $M_{-i}$ at $x_i$, will take the same value in a $\sigma$ neighbourhood of $x_i$ (this is, every rival plays 0 - in $\overline{M}_{-i}$ - or every rival plays 1 - in $\underline{M}_{-i}$ - in the extremal profile). Moreover, as long as there exists an $x_i \in [\underline{X}, \overline{X}]$ such that player $i$ is uncertain about all her rivals play in $M_{-i}$ at a neighbourhood of $x_i$, the intersection is unique and thus the extremal profiles are well defined.

Note that $\forall\ s_{-i} \in M_{-i}$ and $\forall\ x_i \in \left[\underline{X}, \overline{X}\right]$ we have

$$\Delta\Pi_i(\bar{s}_{-i}, x_i) \geq \Delta\Pi_i(s_{-i}, x_i) \geq \Delta\Pi_i\big(\underline{s}_{-i}, x_i\big) \qquad \text{with } \bar{s}_{-i} \in \overline{M}_{-i} \text{ and } \underline{s}_{-i} \in \underline{M}_{-i}.$$

In words, by strategic substitutes (A1) if player $i$, upon receiving a signal $x_i$ and assuming that her opponents are using the strategy profile $\bar{s}_{-i}$ ($\underline{s}_{-i}$), chooses action 0 (resp. 1), then she will choose action 0 (resp. 1) for all $s_{-i} \in M_{-i}$.

In the next section we combine all these objects to describe the process of iterated elimination of strategies.

## 5.2 Iterated Elimination of Strictly Conditionally Dominated Strategies

We now define the process of Iterative Elimination of (interim) Strictly Conditionally Dominated Strategies (IESCDS). Since we have the dominance regions, we know that in any reasonable strategy player $i$ plays 0 when the signal is below $\underline{k}_i$ and plays 1 if the signal is above $\bar{k}_i$. Thus, without loss of generality, for each player $i$ the initial set of conditionally undominated strategies is:

$$\begin{aligned} S_i^0 &:= S_i(\underline{k}_i) \\ &= \left\{s_i \in S_i : s_i(x_i) = 0 \text{ for all } x_i < \underline{k}_i \text{ and } s_i(x_i) = 1 \text{ for all } x_i > \bar{k}_i\right\} \end{aligned} \tag{6}$$

and $S^0 := \times_{j\in\mathcal{N}} S_j^0$.

At each step $t > 0$ the set of conditionally undominated strategies is defined as:

$$S_i^t := S_i\big(\underline{x}_i^t\big) \tag{7}$$

and $S^t := \times_{j\in\mathcal{N}} S_j^t$, where for each player $i \in \mathcal{N}$, $\{\underline{x}_i^t\}_{t=0}^{\infty}$ is a sequence of signals such that $\underline{x}_i^0 = \underline{k}_i$, and for $t > 0$ each element of the sequence is calculated as follows:

- If $\underline{x}_i^{t-1} < x_i^*$ then

$$\underline{x}_i^t := \min\{x_i : \ \Delta\Pi_i(\bar{s}_{-i}, x_i) = 0\}, \tag{8}$$

  with $\bar{s}_{-i} \in \overline{\mathcal{S}}_{-i}\big(\underline{x}^{t-1}\big)$.

- If $\underline{x}_i^{t-1} \geq x_i^*$ then

$$\underline{x}_i^t := \min\{x_i : \ x_i > x_i^* \text{ and } \Delta\Pi_i\big(\underline{s}_{-i}, x_i\big) = 0\}, \tag{9}$$

  with $\underline{s}_{-i} \in \underline{\mathcal{S}}_{-i}\big(\underline{x}^{t-1}\big)$.

Each element of the sequence represents the minimum signal among which player $i$ is indifferent between the two actions, but just considering strategy profiles of her opponents that belong to the set $\mathcal{S}_{-i}\big(\underline{x}^{t-1}\big)$. This is, considering only the strategy profiles that, for some strategy $s_i \in S_i^{t-1} = S_i\big(\underline{x}_i^{t-1}\big)$, consist of strategies that are mutually best responses (excluding player $i$). As is the convention in the literature on optimization, if there is no solution for problem (8) we set $\underline{x}_i^t = x_i^*$ and if there is no solution for problem (9) we set $\underline{x}_i^t = \bar{k}_i$.

It is direct to verify that $\{\underline{x}_i^t\}_{t=0}^{\infty}$ is a non-decreasing sequence and thus $S_i^t \subseteq S_i^{t-1}$, and that $\mathcal{S}_{-i}(\underline{x}^t) \subseteq \mathcal{S}_{-i}\big(\underline{x}^{t-1}\big)$.

In Figures 9 and 10 we illustrate the structure of the surviving strategies for the three player case at $t > 0$. Given $\underline{x}_i^t$, every strategy in $S_i^t$ prescribes action 0 for signals less than the minimum between $\underline{x}_i^t$ and $x_i^*$; and prescribes action 1 for signals in player $i$'s upper dominance region and for signals in the interval $(x_i^*, \underline{x}_i^t)$ (if not empty). Figure 9 shows the case where $\underline{x}_2^t < x_2^*$ and Figure 10 shows the case where $\underline{x}_2^t \geq x_2^*$.

**Definition 5.** *We will say that a strategy $s_i$ survives* IESCDS *if $s_i \in \bigcap_{t=0}^{\infty} S_i^t$.*

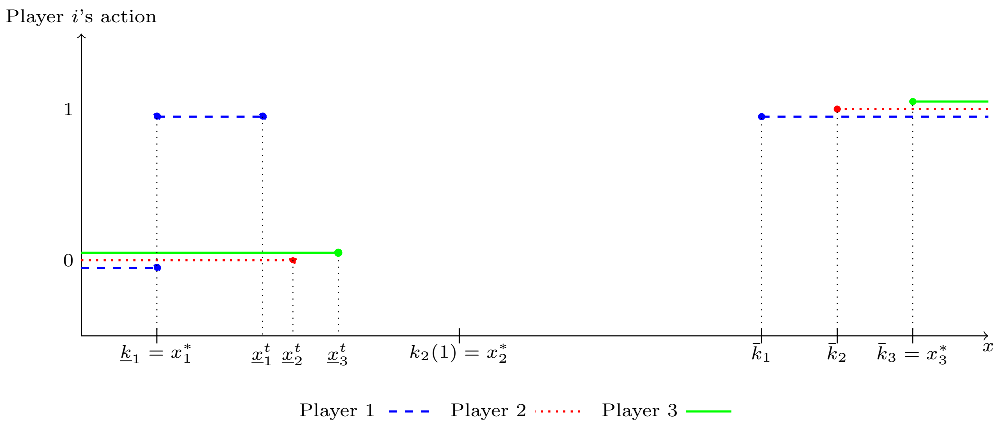


Figure 9: Case $\underline{x}_2^t < x_2^*$

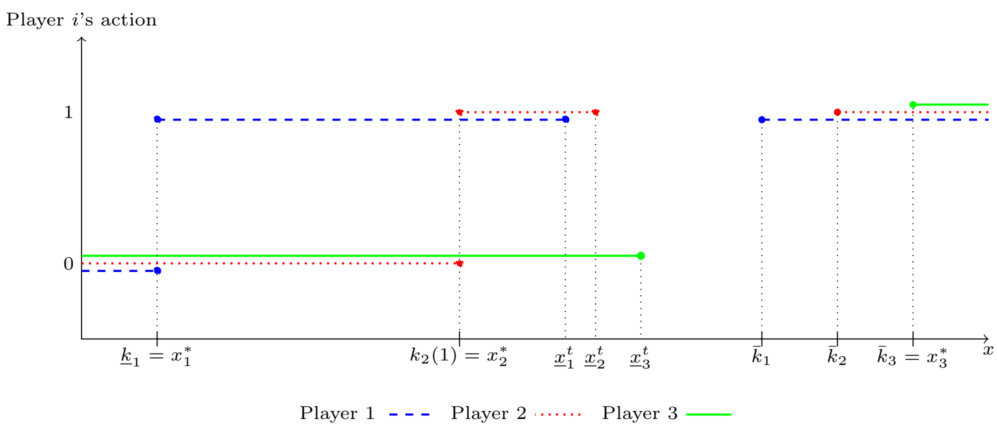


Figure 10: Case $\underline{x}_2^t \geq x_2^*$

### 5.3 Steps of the proof

We now proceed with the proof of the Theorem We develop the proof stating two Lemmata and a Proposition, the proofs of which are relegated to the appendix.

The first Lemma states that $s^*$ is in fact a Bayesian Nash Equilibrium of $\mathrm{G}_{\mathcal{N}}(\sigma)$.

**Lemma 3.** *As* $\sigma \to 0$ *we have* $s^* \in \mathrm{BNE}(\mathrm{G}_{\mathcal{N}}(\sigma))$.

This is clear since if the signal of player $i$ is below the threshold $x_i^* \equiv k_i(i-1)$ and her rivals play $s_{-i}^*$, then she sees at most $i-1$ players playing action 1. Therefore, by definition of $k_i(i-1)$, the optimal action is 0. Analogously, if the signal of player $i$ is greater than $x_i^*$, the optimal action is 1.

The next Lemma establishes a very natural and intuitive property: the process of IESCDS described above does not rule out any Bayesian Nash equilibrium.

**Lemma 4.** $\forall\ t$, $\mathrm{BNE}(\mathrm{G}_{\mathcal{N}}(\sigma)) \subseteq S^t$.

This is clear since for every player $i$ all equilibrium strategy profiles of her rivals belong, by definition, to the set $\mathcal{S}_{-i}\big(\underline{x}^0\big)$ and so equilibrium profiles are not eliminated in the first round. For the same reason, they are never eliminated.

Lemmata 3 and 4 imply that $s^* \in S^t\ \forall\ t \geq 0$ and therefore survives the process of IESCDS.

**Proposition 2.** *As* $\sigma \to 0$*, the profile* $s^*$ *is the only strategy profile that survives the process of* IESCDS.

Since Lemma 3 states that $s^*$ is an equilibrium and Lemma 4 that all equilibria survive the process of IESCDS, Proposition 2 closes the proof of the Theorem.

# 6 Conclusions

The global game approach is a proven method to incorporate more realistic assumptions in game-theoretic models. Assuming a very general payoff structure, the approach examines Nash equilibria as a limit of equilibria of payoff-perturbed games. Carlsson and van Damme (1993) show that in binary action two-player games, there exists a unique equilibrium profile surviving iterated deletion of strictly dominated strategies. Later, this result was generalized by Frankel et al. (2003) to many players and actions, but limiting the analysis to games with strategic complementarities.

Continuing with this line of research, we extend the literature proving an equilibrium selection result for a class of global games with strategic substitutes. Assuming a particular asymmetry in the players' dominance regions, we prove that for a general class of binary action, $N$-player games, each such game has a unique equilibrium strategy profile as the noise goes to 0. This result might allow us to analyze a wide class of games of strategic substitutes such as collective action problems, entry-exit models in industrial organization etc. In particular, we might apply the result to a model of public good provision as is described for example in Section 2. The interesting conclusion to this application is that the equilibrium profile induces an efficient provision of the public good, and the contributions come from the lowest cost contributors. In general the result provides a useful tool for applications.

Further research must be devoted to extend the result to dominance solvability of the unique equilibrium (see Harrison and Jara-Moroni, 2015, for an example of this type of results).

# Appendix

# A Proofs of Lemmata 1 to 4

To ease the exposition, we now repeat the formal statements of Lemmata 1 to 4 before each of their proofs.

**Lemma** (Lemma 1)**.** *For all $i \in \mathcal{N}$ and for all $n \in \{0, \ldots, N-1\}$ there exists a unique $k_i(n) \in \left[\underline{X}, \overline{X}\right]$ solving $\Delta\pi_i(n, k_i(n)) = 0$. Moreover, $\Delta\pi_i(n, x) < 0$, $\forall\ x < k_i(n)$; and $\Delta\pi_i(n, x) > 0$, $\forall\ x > k_i(n)$.*

*Proof.* For any $n$, from assumption A3 (monotonicity of $\Delta\pi_i(n, \cdot)$), if there exists a solution to $\Delta\pi_i(n, \cdot) = 0$, it is unique. Furthermore, by assumptions A1 (SS) and A4 (IP) we have that for any $0 < n < N-1$:

$$\Delta\pi_i(n, \underline{k}_i) < \Delta\pi_i(0, \underline{k}_i) = 0 \qquad\qquad \Delta\pi_i\big(n, \bar{k}_i\big) > \Delta\pi_i\big(N-1, \bar{k}_i\big) = 0.$$

Thus, assumption A2 (C) provides the existence of a point $\tilde{x} \in ]\underline{k}_i, \bar{k}_i[$ such that $\Delta\pi_i(n, \tilde{x}) = 0$. Clearly $\tilde{x} = k_i(n)$.

Again, assumption A3 (M) implies that if $x < k_i(n)$, then $\Delta\pi_i(n, x) < \Delta\pi_i(n, k_i(n)) = 0$; and if $x > k_i(n)$, then $\Delta\pi_i(n, x) > \Delta\pi_i(n, k_i(n)) = 0$.

■

**Lemma** (Lemma 2)**.** *There exists a value $\sigma_0 > 0$ such that $\forall\ \sigma \in (0, \sigma_0)$, if $j > i$ and $x_j - x_i \leq \sigma$, then $\Delta\pi_i(n, x_i) - \Delta\pi_j(n, x_j) > 0\ \forall\ n$.*

*Proof.* From assumption A5 (PO) we know that there exists a players order $\{1, \ldots, N\}$ and a number $\alpha > 0$ such that if $j > i$ then $\Delta\pi_i(n, x) - \Delta\pi_j(n, x) > \alpha$ for any $n$ and any $x$.[14]

Hence from A3 (M), if $x_j - x_i < 0$ monotonicity implies that $\Delta\pi_i(n, x_i) - \Delta\pi_j(n, x_j) > \alpha$.

Moreover, from continuity of $\Delta\pi_i(n, \cdot)$ and compactness of $\left[\underline{X}, \bar{X}\right]$ we know that there exist for each $j > i$ and $n \in \{0, \ldots, N-1\}$, numbers $\sigma_{jin} > 0$ such that if $x_j - x_i \leq \sigma_{jin}$ then $\Delta\pi_i(n, x_i) - \Delta\pi_j(n, x_j) > \alpha$. Let

$$\sigma_0 \equiv \min\left\{\sigma_{jin} : j > i, n \in \{0, \ldots, N-1\}\right\}. \tag{10}$$

Clearly, since the set $\{jin : j > i, n \in \{0, \ldots, N-1\}\}$ is finite, $\sigma_0 > 0$. Then, if $x_j - x_i \leq \sigma_0$ we have $\Delta\pi_i(n, x_i) - \Delta\pi_j(n, x_j) > \alpha > 0$ for all $n$ and any $j > i$.

■

[14] See Footnote 8

**Lemma** (Lemma 3)**.** *As* $\sigma \to 0$ *we have* $s^* \in \mathrm{BNE}(\mathrm{G}_{\mathcal{N}}(\sigma))$.

*Proof.* Define $\sigma_i$ as the distance between player $i$'s switching point in $s_i^*$, $x_i^* = k_i(i-1)$, and the value $k_{i+1}(i-1)$. This is:

$$\sigma_i := k_i(i-1) - k_{i+1}(i-1)\,.$$

And now define $\hat{\sigma}$ as the minimum of the $\sigma_i$'s

$$\hat{\sigma} := \min_{i \in \mathcal{N}} \{\sigma_i\}\,.$$

Consider $\sigma < \hat{\sigma}$ and take a player $i \in \mathcal{N}$. We have to prove that if $x_i < x_i^*$ then $\Delta\Pi_i\big(s_{-i}^*, x_i\big) \leq 0$ and if $x_i > x_i^*$ then $\Delta\Pi_i\big(s_{-i}^*, x_i\big) \geq 0$.

Take a player $j < i$ and consider the interval $]x_j^* - \sigma, x_{j+1}^* - \sigma]$.

If $x_{j+1}^* - \sigma \geq x_i \geq x_j^* + \sigma$ then player $i$ knows that the number of players that are playing action 1 is exactly $j$ (all players less or equal to $j$ are playing 1 and every player greater or equal to $j+1$ - except $i$ - is playing 0). Thus,

$$\Delta\pi_i\left(\sum_{l \neq i} s_l(x_l)\,, x\right) = \Delta\pi_i(j, x)$$

in the support of $dF_{(\sigma,-i)}(\cdot \mid x_i)$. Moreover, since $x_i \leq x_{j+1}^* - \sigma = k_{j+1}(j) - \sigma$ then, from A5 (PO), $x_i \leq k_i(j) - \sigma$ and so

$$\Delta\pi_i(j, x) < 0 \qquad \forall\ x \in \left[x_i - \tfrac{\sigma}{2}, x_i + \tfrac{\sigma}{2}\right]$$

Thus,

$$\Delta\Pi_i\big(s_{-i}^*, x_i\big) = \int_{x_i - \frac{\sigma}{2}}^{x_i + \frac{\sigma}{2}} \int_{x_{-i}} \Delta\pi_i\left(\sum_{j \neq i} s_j(x_j)\,, x\right) dF_{(\sigma,-i)}(x_{-i} \mid x_i)\, dP(x \mid x_i) < 0$$

If $x_j^* - \sigma < x_i \leq x_j^* + \sigma$ then there is a positive probability of player $j$ playing action 0 instead of 1 (all the other players actions remain constant). Therefore $\Delta\Pi_i\big(s_{-i}^*, x_i\big)$ is a weighted average between $\Delta\pi_i(j-1, x)$ and $\Delta\pi_i(j, x)$. However, since

$$\sigma < k_{j+1}(j-1) - k_j(j-1) \leq k_i(j-1) - k_j(j-1)$$

we know that

$$\Delta\pi_i(j-1, x) < 0 \qquad \forall\ x \in \left[x_i - \tfrac{\sigma}{2}, x_i + \tfrac{\sigma}{2}\right].$$

We conclude that in this case we also have

$$\Delta\Pi_i\big(s_{-i}^*, x_i\big) < 0.$$

We illustrate this analysis in Figure 11.

Summarising, for any $j < i$ we have that for all $x_i \in ]x_j^* - \sigma, x_{j+1}^* - \sigma]$, $\Delta\Pi_i\big(s_{-i}^*, x_i\big) < 0$ and so, for all $x_i < x_i^* - \sigma$, $\Delta\Pi_i\big(s_{-i}^*, x_i\big) < 0$.

With analogous arguments, we may prove that for all $x_i > x_i^* + \sigma$, $\Delta\Pi_i\big(s_{-i}^*, x_i\big) > 0$.

Therefore, as $\sigma \to 0$, the strategy $s_i^*$ is a best response to $s_{-i}^*$.

■

**Lemma** (Lemma 4)**.** $\forall\ t$, $\mathrm{BNE}(\mathrm{G}_{\mathcal{N}}(\sigma)) \subseteq S^t$.

*Proof.* Let $s^e \in \mathrm{BNE}(\mathrm{G}_{\mathcal{N}}(\sigma))$.

It is direct to see that $s^e \in S^0$.

For $t > 0$ let us suppose that $s^e \in S^{t-1}$. Then, since it is an equilibrium, we have $\forall\ i \in \mathcal{N}$, $s_{-i}^e \in \mathcal{S}_{-i}\big(\underline{x}^{t-1}\big)$.

Consider e player $i \in \mathcal{N}$. The strategies in $S_i(\underline{x}_i^t)$ prescribe the same action than the strategies in $S_i\big(\underline{x}_i^{t-1}\big)$ for all $x_i < \underline{x}_i^{t-1}$ and all $x_i > \bar{k}_i$. The only discrepancies my appear in the interval $\big(\underline{x}_i^{t-1}, \underline{x}_i^t\big)$. Now, the actions prescribed by strategies in $S_i(\underline{x}_i^t)$ in this interval, are best reply actions even to the worst case scenarios in $\mathcal{S}_{-i}\big(\underline{x}^{t-1}\big)$. Since $s_{-i}^e \in \mathcal{S}_{-i}\big(\underline{x}^{t-1}\big)$ and the action prescribed by $s_i^e$ in this interval is a best reply to $s_{-i}^e$, it must be the one prescribed by the set $S_i(\underline{x}_i^t)$. Thus $s_i^e \in S_i(\underline{x}_i^t)$.

Hence it must be the case that $s^e \in S^t$.

■

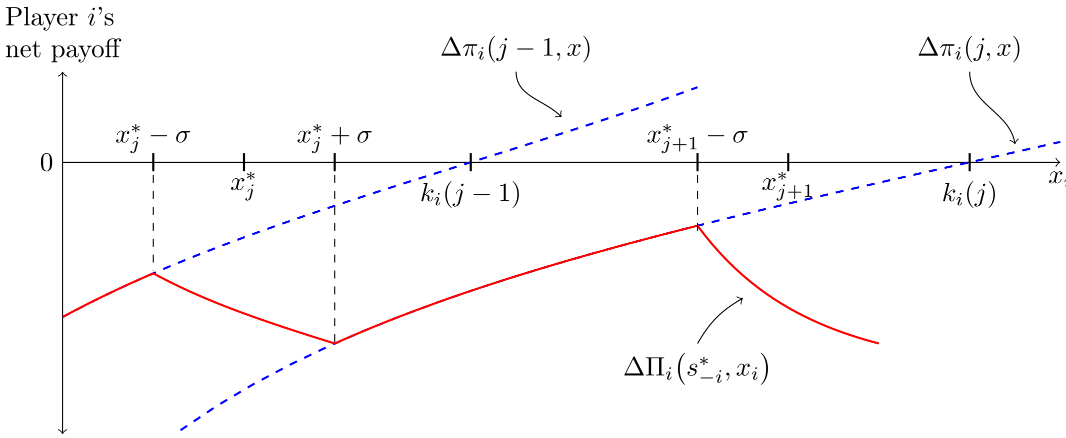


Figure 11: Optimal action against $s_{-i}^*$ as a function of $x_i$, for $x_i < x_i^*$.

# B Proof of Proposition 2

To prove the proposition we will prove that for all $i \in \mathcal{N}$, $\lim_{t\to\infty} \underline{x}_i^t = \bar{k}_i$. Indeed, if this is the case, then for $i \in \mathcal{N} \setminus \{N\}$, any strategy $s_i \in S_i^t$ for sufficiently large $t$ has the form:

$$s_i(x_i) = \begin{cases} 0 & \text{if } x_i < x_i^* \\ 1 & \text{if } x_i^* < x_i < \underline{x}_i^t \text{ or } \bar{k}_i < x_i \end{cases}$$

and for $i = N$ we have

$$s_i(x_i) = \begin{cases} 0 & \text{if } x_i < \underline{x}_i^t \\ 1 & \text{if } \bar{k}_i < x_i \end{cases}$$

and in the limit, the only strategy that satisfies this is $s^*$. Thus, for every player $i \in \mathcal{N}$ the only strategy in $\bigcap_{t=0}^{\infty} S_i^t$ is $s_i^*$.

We prove Proposition 2 by induction on the number of players. We begin then by proving that it is true for the game $G_{\{1,2\}}(\sigma)$.

## B.1 The two player case

Before commencing the proof, let us state some useful remarks.

In the two player case we have two non-decreasing sequences $\{\underline{x}_1^t\}_{t=0}^{\infty}$ and $\{\underline{x}_2^t\}_{t=0}^{\infty}$ that by construction are bounded from above by $\bar{k}_1$ and $\bar{k}_2$ (respectively). The extremal profiles faced by a player are just strategies of her rival, so we call them extremal strategies.

Regarding the sets $S_i(\underline{x}_i^t)$, we know that for all $t \geq 0$, $\underline{x}_1^t \geq \underline{k}_1 = x_1^*$ and so the update of the sequence for player 1 is governed by equation (9) and the set $S_1(\underline{x}_1^t)$ has the following structure:

$$S_1\left(\underline{x}_1^t\right) = \left\{ s_1 \in S_1 :\ s_1(x_1) = 0 \text{ if } x_1 < \underline{k}_1 \text{ and } s_1(x_1) = 1 \text{ if } x_1 \in (\underline{k}_1, \underline{x}_1^t) \cup (\bar{k}_1, \overline{X} + \sigma) \right\}. \qquad (11)$$

For player 2 we know that $\underline{x}_2^t < \underline{k}_2 = x_2^*$ and so the update of the sequence is governed by equation (8) and the set $S_2(\underline{x}_2^t)$ has the structure:

$$S_2\left(\underline{x}_2^t\right) = \left\{ s_2 \in S_2 :\ s_2(x_2) = 0 \text{ if } x_2 < \underline{x}_2^t \text{ and } s_2(x_2) = 1 \text{ if } x_2 > \bar{k}_2 \right\}. \qquad (12)$$

Since the update for player 1 is governed by equation (9), the relevant extremal strategies are the lower extremal strategies, which are the ones that give player 1 the less incentive to play action 1. These, from A1, are the strategies of player 2 that belong to $S_2\left(\underline{x}_2^{t-1}\right)$ and prescribe action 1 in the interval $\left(\underline{x}_2^{t-1}, \bar{k}_2\right)$. Therefore at each $t \geq 1$ there is a unique extremal strategy of player 2 which is the switching strategy on $\underline{x}_2^{t-1}$:

$$\underline{s}_2^{t-1}(x_2) := \hat{s}_2\left(x_2; \underline{x}_2^{t-1}\right), \tag{13}$$

where $\hat{s}_2\left(\cdot; \underline{x}_2^{t-1}\right)$ is defined in (4).

Analogously, since the update for player 2 is governed by equation (8), the relevant extremal strategies are the upper extremal strategies, which are the ones that give player 2 the less incentive to play action 0 (equivalently, the most incentive to play action 1). These, in turn, are the strategies of player 1 that belong to $S_1\left(\underline{x}_1^{t-1}\right)$ and prescribe action 0 in the interval $\left(\underline{x}_1^{t-1}, \bar{k}_1\right)$. Therefore at each $t \geq 1$ there is a unique extremal strategy which is the strategy:

$$\bar{s}_1^{t-1}(x_1) := \begin{cases} 0 & \text{if } x_1 < \underline{k}_1 \\ 1 & \text{if } \underline{k}_1 < x_1 < \underline{x}_1^{t-1} \\ 0 & \text{if } \underline{x}_1^{t-1} < x_1 < \bar{k}_1 \\ 1 & \text{if } \bar{k}_1 < x_1 \end{cases} \tag{14}$$

We now turn to the analysis of the process. Let us define $\bar{\sigma} \equiv \min\left\{(\underline{k}_2 - \underline{k}_1), (\bar{k}_2 - \bar{k}_1), \sigma_0\right\}$, where $\sigma_0$ is defined in equation (10) in the proof of Lemma 2, and take $0 < \sigma < \bar{\sigma}$.

As long as $\underline{x}_i^t < \bar{k}_i$, for $t > 0$ the sequences $\{\underline{x}_1^t\}_{t=0}^{\infty}$ and $\{\underline{x}_2^t\}_{t=0}^{\infty}$ satisfy:

$$|\underline{x}_1^t - \underline{x}_2^t| < \sigma \tag{15}$$

Since they are bounded from above, there exist limit points $\underline{x}_1^{\infty} \leq \bar{k}_1$ and $\underline{x}_2^{\infty} \leq \bar{k}_2$.

By construction $\Delta\Pi_1\left(\underline{s}_2^{t-1}, \underline{x}_1^t\right) = 0$ and $\Delta\Pi_2\left(\bar{s}_1^{t-1}, \underline{x}_2^t\right) = 0$ where the strategies $\underline{s}_2^{t-1}$ and $\bar{s}_2^{t-1}$ are defined in equations (13) and (14), respectively. From (15) we know that $\underline{x}_1^t$ would reach $\bar{k}_1$ before $\underline{x}_2^t$ reaches $\bar{k}_2$.

If there exists $t$ such that $\underline{x}_1^t = \bar{k}_1$ then $\bar{s}_1^t = s_1^*$. Thus, for $x_2 > \underline{k}_2 + \sigma$ we get

$$\begin{aligned} \Delta\Pi_2\left(\bar{s}_1^t, x_2\right) &= \Delta\Pi_2(s_1^*, x_2) \\ &= \Delta\pi_2(1, x_2) \end{aligned}$$

and so $\underline{x}_2^t = \bar{k}_2$ which is what we wanted to prove.

If not, then either $\underline{x}_1^{\infty} < \bar{k}_1$ or $\underline{x}_1^{\infty} = \bar{k}_1$ and $\underline{x}_1^t < \bar{k}_1, \forall\, t > 0$.

If $\underline{x}_1^{\infty} < \bar{k}_1$ then, considering the equivalence $F_\sigma(x_1 \mid x_2) = 1 - F_\sigma(x_1 \mid x_2)$, we have

$$\begin{aligned} \Delta\Pi_1\left(s_2^{t-1}, \underline{x}_1^t\right) &= \Delta\pi_1(1, \underline{x}_1^t) F_\sigma(\underline{x}_1^t \mid \underline{x}_2^{t-1}) + \Delta\pi_1(0, \underline{x}_1^t)(1 - F_\sigma(\underline{x}_1^t \mid \underline{x}_2^{t-1})) \\ \Delta\Pi_2\left(s_1^t, \underline{x}_2^t\right) &= \Delta\pi_2(0, \underline{x}_2^t)\left(1 - F_\sigma(\underline{x}_1^t \mid \underline{x}_2^t)\right) F_\sigma\left(\bar{k}_1 \mid \underline{x}_2^t\right) + \\ &\quad \Delta\pi_2(1, \underline{x}_2^t)\left[1 - \left(1 - F_\sigma(\underline{x}_1^t \mid \underline{x}_2^t)\right) F_\sigma\left(\bar{k}_1 \mid \underline{x}_2^t\right)\right] \end{aligned}$$

thus, the conditions are

$$\begin{aligned} 0 &= \Delta\pi_1(1, \underline{x}_1^t) F_\sigma(\underline{x}_1^t \mid \underline{x}_2^{t-1}) + \Delta\pi_1(0, \underline{x}_1^t)(1 - F_\sigma(\underline{x}_1^t \mid \underline{x}_2^{t-1})) \\ 0 &= \Delta\pi_2(0, \underline{x}_2^t)\left(1 - F_\sigma(\underline{x}_1^t \mid \underline{x}_2^t)\right) F_\sigma\left(\bar{k}_1 \mid \underline{x}_2^t\right) + \\ &\quad \Delta\pi_2(1, \underline{x}_2^t)\left[1 - \left(1 - F_\sigma(\underline{x}_1^t \mid \underline{x}_2^t)\right) F_\sigma\left(\bar{k}_1 \mid \underline{x}_2^t\right)\right]. \end{aligned} \tag{16}$$

Then, taking limit when $t \to \infty$ we get

$$\begin{aligned} 0 &= \Delta\pi_1(1, \underline{x}_1^{\infty}) F_\sigma(\underline{x}_1^{\infty} \mid \underline{x}_2^{\infty}) + \Delta\pi_1(0, \underline{x}_1^{\infty})(1 - F_\sigma(\underline{x}_1^{\infty} \mid \underline{x}_2^{\infty})) \\ 0 &= \Delta\pi_2(0, \underline{x}_2^{\infty})(1 - F_\sigma(\underline{x}_1^{\infty} \mid \underline{x}_2^{\infty})) F_\sigma\left(\bar{k}_1 \mid \underline{x}_2^{\infty}\right) + \\ &\quad \Delta\pi_2(1, \underline{x}_2^{\infty})\left[1 - (1 - F_\sigma(\underline{x}_1^{\infty} \mid \underline{x}_2^{\infty})) F_\sigma\left(\bar{k}_1 \mid \underline{x}_2^{\infty}\right)\right]. \end{aligned}$$

Multiplying the first equation by $F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty) > 0$ [15] and subtracting the second one we get

$$\begin{aligned}[\Delta\pi_1(1,\underline{x}_1^\infty) - \Delta\pi_2(1,\underline{x}_2^\infty)]\, F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty) F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty) +& \\ [\Delta\pi_1(0,\underline{x}_1^\infty) - \Delta\pi_2(0,\underline{x}_2^\infty)]\,(1 - F_\sigma(\underline{x}_1^\infty \mid \underline{x}_2^\infty)) F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty)& \\ -\Delta\pi_2(1,\underline{x}_2^\infty) F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty) = 0.&\end{aligned} \tag{17}$$

From Lemma 2 we get that each term in the left hand side of equation (17) is positive and since $\underline{x}_2^\infty < \bar{k}_2$ the last term is strictly positive, which gives us a contradiction.

Thus, the only remaining option is that $\underline{x}_1^\infty = \bar{k}_1$ and $\underline{x}_1^t < \bar{k}_1$, $\forall\, t > 0$.

If this is the case, the update equations (16) are still valid and taking limit when $t \to \infty$ we get

$$\begin{aligned} 0 &= \Delta\pi_1(1,\bar{k}_1) F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty) + \Delta\pi_1(0,\bar{k}_1)(1 - F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty)) \\ 0 &= \Delta\pi_2(0,\underline{x}_2^\infty)(1 - F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty)) F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty) + \\ &\qquad \Delta\pi_2(1,\underline{x}_2^\infty)\left[1 - (1 - F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty)) F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty)\right]. \end{aligned}$$

By definition, $\Delta\pi_1(1,\bar{k}_1) = 0$ and so from the first equation we have that $F_\sigma(\bar{k}_1 \mid \underline{x}_2^\infty) = 1$. Replacing in the second equation, we get that $\Delta\pi_2(1,\underline{x}_2^\infty) = 0$ which in turn implies that $\underline{x}_2^\infty = \bar{k}_2$ which can not be since $\sigma < \bar{k}_2 - \bar{k}_1$.

We conclude then that it must be the case that there exists $t$ such that $\underline{x}_1^t = \bar{k}_1$ and $\underline{x}_2^t = \bar{k}_2$.

## B.2 The N-player case

In the previous section what we actually proved is that in the game $G_{\{1,2\}}(\sigma)$ there exists a finite $t$ such that $\underline{x}_i^t = \bar{k}_i = k_i(1)$ for both players $i \in \{1,2\}$.

For the $N$-player case, since we will use induction on the number of players, we will assume that in the game $\mathrm{G}_{\mathcal{N}-1}(\sigma)$ there exists a finite $t$ such that $\underline{x}_i^t = k_i(N-2)$ for all players $i \in \mathcal{N} - 1 = \{1, \ldots, N-1\}$. We will have to prove that in a finite $t$ the sequences reach the limit of the upper dominance region in $\mathrm{G}_{\mathcal{N}}(\sigma)$, $k_i(N-1)$.

In order to develop the proof using an inductive argument it is first necessary to formally introduce the notion of *Reduced Game*.

**Definition 6** (Reduced Game)**.** *Consider an incomplete information game* $\mathrm{G}_{\mathcal{N}}(\sigma)$ *as defined in Section 3.1, and an arbitrary subset of players* $I \subseteq \mathcal{N}$*. Let* $s_I = (s_i)_{i \in I}$ *and* $s_{-I} = (s_i)_{i \notin I}$*. Conditionally on* $s_{-I}$*, we define the* Reduced Game $\Gamma_I(\sigma \mid s_{-I})$ *(reduced from* $N$ *players to* $\#I$ *players) of the original game* $\mathrm{G}_{\mathcal{N}}(\sigma)$ *as the game with the same structure as* $\mathrm{G}_{\mathcal{N}}(\sigma)$ *where every player not in* $I$ *sticks to the strategy prescribed by the profile* $s_{-I}$*.*

It is easy to check that if $\mathrm{G}_{\mathcal{N}}(\sigma)$ satisfies assumptions A1, A2, A3 and A5, the same assumptions hold for the reduced game $\Gamma_I(\sigma \mid s_{-I})$. Moreover, if conditionally on $s_{-I}$ there exists an interval of signals $[\underline{\lambda}, \overline{\lambda}] \subseteq [\underline{X}, \overline{X}]$ such that for every player $i \in I$, there exist upper and lower dominance regions (according to assumption A4), then $\Gamma_I(\sigma \mid s_{-I})$ is a reduced game that holds the same properties of the original game $\mathrm{G}_{\mathcal{N}}(\sigma)$. Moreover, note that the reduced game $\Gamma_{\mathcal{N}-1}(\sigma \mid s_N^*)$, where player $N$'s strategy is the switching strategy $\hat{s}_N(\cdot; x_N^*)$, is equivalent to the game $\mathrm{G}_{\mathcal{N}-1}(\sigma)$. These facts may allow us to use results from games with less players.

Recall that in $s^*$ each player $i$ switches from 0 to 1 in $x_i^* = k_i(i-1)$ and so the equilibrium strategy $s_i^*$ does not depend on the number of players.[16] When we pass from a game with $N$ players to a game with $N-1$ players, what changes is the limits of the upper dominance regions (they are $k_i(N-1)$ in the $N$-player game and $k_i(N-2)$ in the $N-1$-player game). Thus, in the $N$-player game the switching point of player $N-1$ is $x_{N-1}^* = k_{N-1}(N-2) \neq \bar{k}_{N-1} = k_{N-1}(N-1)$. In the $N-1$ player game, we do have $x_{N-1}^* = \bar{k}_{N-1} = k_{N-1}(N-2)$.

We now turn to the analysis of the process. Let us define $\bar{\sigma} \equiv \min\left\{(\underline{k}_i - \underline{k}_{i-1})_{i=2}^N, (\bar{k}_i - \bar{k}_{i-1})_{i=2}^N, \sigma_0\right\}$, where $\sigma_0$ is defined in equation (10) in the proof of Lemma 2, and take $0 < \sigma < \bar{\sigma}$.

By definition of $\sigma$, we know that $\underline{k}_{N-1} + \sigma < \underline{k}_N$, then[17] $\forall\, i = \{1, \ldots, N-1\}$ and $\forall\, x_i < \underline{k}_N - \sigma$ player $i$'s payoff is equal to $\Delta\Pi_i\big(s_{-\{i,N\}}, s_N^*, x_i\big)$. So at least the $N-1$ first rounds of elimination

[15] From (15) and the assumption $\underline{x}_1^\infty < \bar{k}_1$ we get that $\underline{x}_2^\infty \leq \underline{x}_1^\infty + \sigma < \bar{k}_1 + \sigma$.

[16] The player that is in the $i$-th position (player $i$) determines her switching point depending only on her position in the set of players and not on how many players interact with her.

[17] Recall that $\underline{k}_{N-1} > \underline{k}_{N-2} > \ldots > \underline{k}_1$

of $\mathrm{G}_{\mathcal{N}}(\sigma)$ are equal to the $N-1$ first rounds of elimination of the reduced game $\Gamma_{\mathcal{N}-1}(\sigma \mid s_N^*)$ (see Definition 6), which in turn es equal to $\mathrm{G}_{\mathcal{N}-1}(\sigma)$. Note that as long as the sequences $\underline{x}_i^t$, $i \in \mathcal{N}-1$ do not reach the threshold $\underline{k}_N - \sigma$, each step of the process in $\mathrm{G}_{\mathcal{N}}(\sigma)$ is equivalent to a step in $\mathrm{G}_{\mathcal{N}-1}(\sigma)$. By the induction hypothesis, we know that in $\mathrm{G}_{\mathcal{N}-1}(\sigma)$, $\underline{x}_{N-1}^t$ reaches $k_{N-1}(N-2)$ in a finite number of steps. So, necessarily, it must be the case that in $\mathrm{G}_{\mathcal{N}}(\sigma)$, $\underline{x}_{N-1}^t$ reaches $\underline{k}_N - \sigma < k_{N-1}(N-2)$. Let us call $\tilde{t}$, the moment when the first sequence $\underline{x}_i^t$ reaches $\underline{k}_N - \sigma$ (we know that $\underline{x}_N^{\tilde{t}} = \underline{k}_N$).

Note as well that as long as the sequences $\underline{x}_i^t$ that reach $\underline{k}_N - \sigma$, have not surpassed the corresponding player $i$'s switching point, the process for the first $N-1$ is equivalent to that in game $\mathrm{G}_{\mathcal{N}-1}(\sigma)$ and for player $N$ nothing changes, thus we may assume that $\tilde{t}$ is the moment when the first sequence that has surpassed the corresponding player's switching point, reaches $\underline{k}_N - \sigma$.

At any step $t$ of the process, we may partition the set $\mathcal{N}$ into two sets: one set containing those players $i$ who have reached their equilibrium switching point (i.e. $x_i^* \leq \underline{x}_i^{t-1}$) and those players $j$ who have not (i.e. $\underline{x}_j^{t-1} < x_j^*$). Moreover, since the switching points are ordered and $\sigma$ is small, we know that these two sets of players consist of the $l$ first players (those who reached their switching point) and the $N-l$ last players (those who have not), for some $l \in \mathcal{N}$. By construction of the process, player $l+1$'s sequence in step $t$ may at most reach her switching point. If this is the case, then in the next step she will then be part of the set of players that reached their switching point (now the first $l+1$ players).

Regarding the sets $S_i(\underline{x}_i^t)$, we know that for all $i \leq l$, $\underline{x}_i^t \geq x_i^*$ and so the update of the sequence for player $i$ is governed by equation (9) and the set $S_i(\underline{x}_i^t)$ has the following structure:

$$S_i\big(\underline{x}_i^t\big) = \big\{ s_i \in S_i : \; s_i(x_i) = 0 \text{ if } x_i < x_i^* \text{ and } s_i(x_i) = 1 \text{ if } x_i \in (x_i^*, \underline{x}_i^t) \cup (\bar{k}_i, \overline{X} + \sigma) \big\}. \tag{18}$$

For players $i > l$ we know that $\underline{x}_i^t < x_i^*$ and so the update of the sequence is governed by equation (8) and the set $S_i(\underline{x}_i^t)$ has the structure:

$$S_i\big(\underline{x}_i^t\big) = \big\{ s_i \in S_i : \; s_i(x_i) = 0 \text{ if } x_i < \underline{x}_i^t \text{ and } s_i(x_i) = 1 \text{ if } x_i > \bar{k}_i \big\}. \tag{19}$$

Let us first consider a player $i > l$. Since the update for these players is governed by equation (8), the relevant extremal profiles are the upper extremal profiles, which are the ones that give these players the less incentive to play action 0. These, in turn, by strategic substitutes, are the profiles of the other players that belong to the set $\prod_{j \neq i} S_j\big(\underline{x}_j^{t-1}\big)$ that prescribe action 0 in the intervals $\big(\underline{x}_j^{t-1}, \bar{k}_j\big)$, subject to being mutually best responses (profiles of the rivals that prescribe the most zeros, within the mutually best responses). In the worst case, the component of the extremal profile $\bar{s}_{-i}^{t-1}$, associated to the $N^{th}$ player is the equilibrium strategy $s_N^*$. If this is the case, then the update for player $i$ is equal to the update in the reduced game $\Gamma_{\mathcal{N}-1}(\sigma \mid s_N^*) = \mathrm{G}_{\mathcal{N}-1}(\sigma)$. If not, then the new term $\underline{x}_i^t$ would be gretaer than the one obtained in the update in the reduced game. In any case, in the original game the sequence advances further away from $\underline{k}_i$ than in the reduced game.

Second, let us consider a player $i \leq l$. Since the update for player $i \leq l$ is governed by equation (9), the relevant extremal profiles are the lower extremal profiles, which are the ones that give player $i$ the less incentive to play action 1. These, in turn, by strategic substitutes, are the profiles of the other players that belong to the set $\prod_{j \neq i} S_j\big(\underline{x}_j^{t-1}\big)$ that prescribe action 1 in the intervals $\big(\underline{x}_j^{t-1}, \bar{k}_j\big)$, subject to being mutually best responses (profiles of the rivals that prescribe the most "ones", within the mutually best responses). Nevertheless we will see below that the component of the extremal profile $\underline{s}_{-i}^{t-1}$, associated with player $N$ is $s_N^*$ as well.

Indeed, suppose that the strategy of player $N$ in an extremal profile $\underline{s}_{-i} \in \underline{\mathcal{S}}_{-i}\big(\underline{x}^{t-1}\big)$ is different from $s_N^*$. This is, there exists an open interval $O \subset ]\underline{x}_N^{t-1}, \bar{k}_N[$ such that the strategy $s_N \in S_N\big(\underline{x}_N^{t-1}\big)$ in the extremal profile $\underline{s}_{-i}$ prescribes $s_N(x_N) = 1$ for $x_N \in O$. For signals less than $x_{N-1}^*$ we cannot have the $N-1$ dimensional profile $(1, \ldots, 1)$ on $O$, since this profile of actions do not constitute mutually best responses. Therefore, for every $\underline{s}_{-i} \in \underline{\mathcal{S}}_{-i}\big(\underline{x}^{t-1}\big)$ there must be a player $j \neq i$ such that the $j$-th component of $\underline{s}_{-i}$ prescribes action 0 in $O$. By anonymity, we may assume that this player is player $N$.

Therefore, the update for player $i \leq l$ in game $\mathrm{G}_{\mathcal{N}}(\sigma)$ is equivalent to the update in the reduced game $\Gamma_{\mathcal{N}-1}(\sigma \mid s_N^*) = \mathrm{G}_{\mathcal{N}-1}(\sigma)$.

We conclude that, as long as at $t-1$ the sequences are less than $x_{N-1}^*$, the update of every player $i < N$ in the game $\mathrm{G}_{\mathcal{N}}(\sigma)$ sends the next term of the sequence at least as far as in the update in the reduced game $\Gamma_{\mathcal{N}-1}(\sigma \mid s_N^*) = \mathrm{G}_{\mathcal{N}-1}(\sigma)$. For player $N$, the updates of all the other players induce an update of $\underline{x}_N^t > \underline{x}_N^{t-1}$.

Therefore, by the induction hypothesis, for every player $i < N$ there must be a finite $t$ such that, $\underline{x}_i^t$ is equal to the lower bound of the upper dominance region in the game $\mathrm{G}_{\mathcal{N}-1}(\sigma)$. This is, $\forall\, i < N$ there exists $t$ such that, $\underline{x}_i^t = k_i(N-2)$.

Moreover, calling $\tilde{t}$ the step where $\underline{x}_{N-1}^{\tilde{t}} = k_{N-1}(N-2)$, note that for a player for $i < N-1$, if $\underline{x}_i^{t-1} \geq k_i(N-2)$ and $\underline{x}_{N-1}^{t-1} \leq k_{N-1}(N-2)$ then the update for player $i$ in game $\mathrm{G}_{\mathcal{N}}(\sigma)$ is equivalent to the update in the reduced game $\Gamma_{\mathcal{N}-1}(\sigma \mid s_N^*) = \mathrm{G}_{\mathcal{N}-1}(\sigma)$ and so action 1 is strictly dominant for this player (since player $i$ sees at most $N-2$ players taking action 1). So, we can conclude, that at $\tilde{t}$ we have $\underline{x}_{N-1}^{\tilde{t}} = k_{N-1}(N-2)$ and for every player $i < N-1$, $\underline{x}_i^{\tilde{t}}$ is in a $\sigma$-neighborhood of $k_{N-1}(N-2) = x_{N-1}^*$. Again, for player $N$, at each such $t$, the updates of all the other players induce an update of $\underline{x}_N^t$ and so we also have $\underline{x}_N^{\tilde{t}}$ in a $\sigma$-neighborhood of $k_{N-1}(N-2) = x_{N-1}^*$. We illustrate this situation for the 3-player case in Figure 12.

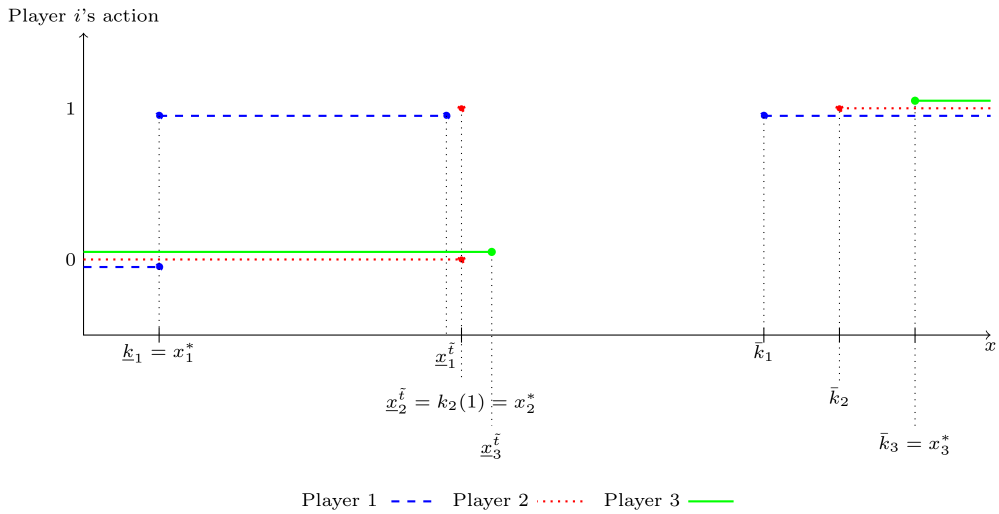


Figure 12: Situation at $\tilde{t}$ for the 3-player case.

We now describe the process for $t > \tilde{t}$ and conclude the proof. Let us assume that for $t-1$ we have[18]

$$\underline{x}_{N-1}^{t-1} \geq x_{N-1}^* = k_{N-1}(N-2)\,.$$

We will follow the same argument described and used above. In this situation we know that for all players $i \leq N-1$, $\underline{x}_i^{t-1} \geq x_i^*$. The update for these players is governed by equation (9) and so the relevant extremal profiles are the lower extremal profiles, which are the ones that give the less incentive to play action 1. These, in turn, by strategic substitutes, are the profiles of the other players that belong to the set $\prod_{j \neq i} S_j\big(\underline{x}_j^{t-1}\big)$ that prescribe action 1 in the intervals $\big(\underline{x}_j^{t-1}, \bar{k}_j\big)$, subject to being mutually best responses.

Consider a player $i \in \{2, \ldots, N-1\}$. In the worst case, the component of the extremal profile $\underline{s}_{-i}^{t-1}$, associated to the $1^{st}$ player is the equilibrium strategy $s_1^*$. If this is the case, then the update for player $i$ is equal to the update in the reduced game $\Gamma_{\{2,\ldots,N\}}(\sigma \mid s_1^*)$ which is an $N-1$ player game that satisfies assumptions A1 to A5. If $s_1^*$ is not the component of the extremal profile $\underline{s}_{-i}^{t-1}$, associated to the $1^{st}$ player, then the new term $\underline{x}_i^t$ would be greater than the one obtained in the update in the reduced game. In any case, in the original game the sequence advances further away from $\underline{k}_i$ than in the reduced game.

The update for player $N$ is governed by equation (8). The relevant extremal profiles are the upper extremal profiles, which are the ones that give less incentive to play action 0. These, in turn, by strategic substitutes, are the profiles of the other players that belong to the set $\prod_{j \neq N} S_j\big(\underline{x}_j^{t-1}\big)$ that prescribe action 0 in the intervals $\big(\underline{x}_j^{t-1}, \bar{k}_j\big)$, subject to being mutually best responses. We will see

[18] It is true for $t-1 = \tilde{t}$.

below that the component of the extremal profile $\underline{s}_{-N}^{t-1}$, associated with player 1 is $s_1^*$ as for players in the set $\{2, \ldots, N-1\}$.

Indeed, suppose that the strategy of player 1 in an extremal profile $\underline{s}_{-N} \in \underline{\mathcal{S}}_{-N}(\underline{x}^{t-1})$ is different from $s_1^*$. This is, there exists an open interval $O \subset ]\underline{x}_1^{t-1}, \bar{k}_1[$ such that the strategy $s_1 \in S_1(\underline{x}_1^{t-1})$ in the extremal profile $\underline{s}_{-N}$ prescribes $s_1(x_1) = 0$ for $x_1 \in O$. However, for signals greater or equal to $x_{N-1}^*$ we cannot have the $N-1$ dimensional profile $(0, \ldots, 0)$ on $O$, since this profile of actions do not constitute mutually best responses. Therefore, for every $\underline{s}_{-N} \in \underline{\mathcal{S}}_{-N}(\underline{x}^{t-1})$ there must be a player $j \neq N$ such that the $j$-th component of $\underline{s}_{-N}$ prescribes action 1 in $O$. By anonymity, we may assume that this player is player 1.

We conclude that the update of every player $i > 1$ in the game $\mathrm{G}_{\mathcal{N}}(\sigma)$ sends the next term of the sequence at least as far as in the update in the reduced game $\Gamma_{\{2,\ldots,N\}}(\sigma \mid s_1^*)$. For player 1, the updates of all the other players induce an update of $\underline{x}_1^t > \underline{x}_1^{t-1}$.

Therefore, by the induction hypothesis, for every player $i > 1$ there must be a finite $t$ such that, $\underline{x}_i^t$ is equal to the lower bound of the upper dominance region in the game $\Gamma_{\{2,\ldots,N\}}(\sigma \mid s_1^*)$. These lower bounds, are the exact same lower bounds of the upper dominance regions of the game $\mathrm{G}_{\mathcal{N}}(\sigma)$. This is, $\forall\ i > 1$ there exists $t$ such that, $\underline{x}_i^t = k_i(N-1) = \bar{k}_i$. If every player but player 1 has reached the upper dominance region, then in the next step, player 1 reaches her upper dominance region.

This completes the proof.

# References


Carlsson, H. and E. van Damme (1993, September). Global games and equilibrium selection. *Econometrica 61*(5), 989–1018.

Frankel, D. M., S. Morris, and A. Pauzner (2003, January). Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory 108*(1), 1–44.

Goldstein, I. and A. Pauzner (2005). Demand-deposit contracts and the probability of bank runs. *The Journal of Finance 60*(3), 1293–1327.

Harrison, R. and P. Jara-Moroni (2015). A dominance solvable global game with strategic substitutes. *Journal of Mathematical Economics 57*, 1–11.

Harsanyi, J. (1973). Games with randomly disturbed payoffs: A new rationale for mixed-strategy equilibrium points. *International Journal of Game Theory 2*, 1–25.

Morris, S. and H. S. Shin (1998, June). Unique equilibrium in a model of self-fulfilling currency attacks. *American Economic Review 88*(3), 587–597.

Morris, S. and H. S. Shin (2003). Global games: Theory and applications. In M. Dewatripont, L. Hansen, and S. Turnovsky (Eds.), *Advances in Economics and Econometrics: Theory and Applications: Eighth World Congress Vol I*, Volume 1, pp. 57–114. Cambridge University Press.

Selten, R. (1975). A reexamination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory 4*, 25–55.